

Mass and Charge in Brane-World and Non-Compact Kaluza-Klein Theories in 5 Dim

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Abstract

In classical Kaluza-Klein theory, with compactified extra dimensions and without scalar field, the rest mass as well as the electric charge of test particles are constants of motion. We show that in the case of a large extra dimension this is no longer so. We propose the Hamilton-Jacobi formalism, instead of the geodesic equation, for the study of test particles moving in a five-dimensional background metric. This formalism has a number of advantages: (i) it provides a clear and invariant definition of rest mass, without the ambiguities associated with the choice of the parameters used along the motion in $5D$ and $4D$, (ii) the electromagnetic field can be easily incorporated in the discussion, and (iii) we avoid the difficulties associated with the “splitting” of the geodesic equation. For particles moving in a general $5D$ metric, we show how the effective rest mass, as measured by an observer in $4D$, varies as a consequence of the large extra dimension. Also, the fifth component of the momentum changes along the motion. This component can be identified with the electric charge of test particles. With this interpretation, both the rest mass and the charge vary along the trajectory. The constant of motion is now a combination of these quantities. We study the cosmological variations of charge and rest mass in a five-dimensional bulk metric which is used to embed the standard $k = 0$ FRW universes. The time variations in the fine structure “constant” and the Thomson cross section are also discussed.

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1 Introduction

Because of the cylindricity condition, in classical or *compactified* Kaluza-Klein theory, the “new” physics predicted by the five-dimensional equations, resides completely in the four potentials of electromagnetism A_μ and the scalar potential Φ . The $4D$ part of the five-dimensional equations can be manipulated to reproduce the four-dimensional physics with some new terms, or corrections. If the electromagnetic potentials are absent and the scalar field is constant, then these new terms vanish identically.

However, the cylindricity condition is not required, nor, in general, sustained. The possibility that our world might contain more than four *non-compact* dimensions has attracted the interest of a great number of researchers.

In higher-dimensional gravity theories, inspired by string theories, instead of cylindricity the matter fields are assumed to be confined to our $4D$ spacetime, which is modeled as a singular hypersurface or “brane” embedded in a larger $(4 + d)$ world. In these theories gravity is a multidimensional interaction that can propagate in the extra d dimensions as well [1]-[3]. In the RS scenario, for $d = 1$, the usual general relativity in $4D$ can be recovered even when the extra dimension is infinite in size [4]-[5].

In space-time-matter theory (STM), inspired by the unification of matter and geometry, instead of cylindricity the matter fields in our $4D$ spacetime are assumed to be “induced” or derived from pure geometry in five dimensions [6]-[7]. Our $4D$ spacetime can, in principle, be *any* four-dimensional hypersurface orthogonal to the extra dimension, not necessarily a singular one [8]-[9].

The existence of a large extra dimension is intriguing and provides a wealth of new physics, even in the absence of electromagnetic field and constant scalar potential. This is illustrated by the effective equations for gravity in $4D$, which predict five-dimensional local and non-local corrections to the usual general relativity in $4D$ [10]-[18]. Also, the geodesic equation for test particles in $5D$ predicts an effective four-dimensional equation of motion with an extra non-gravitational force [19]-[25]. This force, called sometimes “fifth” force, has a component that is a direct consequence of the variation of the rest mass of test particles, induced by the large extra dimension.

In this work we study in more detail the effects of a large extra dimension on the rest mass and electric charge of test particles. The possibility that these quantities might be variable has important implications for the foundations of physics because variable mass and/or charge imply time-varying Thomson cross section $\sigma = (8\pi/3)(q_e^2/m_0c^2)^2$ for the scattering of electromagnetic radiation by a particle of charge q_e and mass m_0 . Also the variation of electric charge q_e implies the variation of the electromagnetic fine structure “constant” $\alpha_{em} = q_e^2/(4\pi\hbar c)$. The latter has attracted considerable attention in view of the recent observational evidence that α_{em} might vary over cosmological time scales [26]-[28]. This, of course, requires the time variation of at least one of the “constants” (q_e , \hbar and c). However, recently a number of theories attribute the variation of the fine structure constant to changes in the fundamental electron charge and preserve c (Lorentz invariance) and \hbar as constants [29]-[33].

We have other theoretical motivations for this study. Namely, that in compactified Kaluza-Klein theory, with the imposition of the so-called cylinder condition, the electric charge is strictly constant. Therefore, we would like to see whether a large extra dimension would once more give new physics and change the classical scenario.

Besides, we have some technical motivations. Firstly, previous results regarding the variation of rest mass of test particles were obtained from the analysis of the geodesic equation [19]-[25], [34]-[37]. However, the mass of the test particle appears nowhere in this equation. Secondly, the interpretation of the rest mass in terms of the extra coordinate, based on the geodesic equation, seems to depend heavily on the choice of system of coordinates and the choice of parameter used to characterize the motion [21], [34], [38].

What we propose here is to define the properties of test particles, as measured by an observer in $4D$, through the Hamilton-Jacobi equation, instead of the geodesic one.

This approach has several advantages. From a physical viewpoint, it provides a clear and invariant defi-

nition of rest mass, without the problems associated with the parameter used along the motion. In addition the electromagnetic field can be easily incorporated in the discussion. From a mathematical viewpoint, this is due to the fact that the Hamilton-Jacobi equation is a *scalar* equation. We thus avoid the complications associated with the $(4+1)$ “splitting ” of the $5D$ geodesic equation¹.

In this work we obtain the general expression for the change of the rest mass of a test particle, as perceived by an observer in $4D$. It includes the combined effects from a large extra dimension, and those from the electromagnetic and scalar potentials.

We also show that the fifth component of the momentum, which in $4D$ can be identified with the electric charge of the test particle, changes along the motion. This effect has no counterpart in classical theory, where the charge is strictly constant. Thus, a large extra dimension leads naturally to a variation of the fine structure constant. For the case where c and \hbar are taken to be constants, we present the equation for the relative variation of α_{em} .

The conservation equation for mass and charge, as well as the relationship in their variation, is also discussed.

2 The Formalism

In this section we give a brief review of the Kaluza-Klein equations in terms of local basis vectors, which are used to project five-dimensional quantities onto spacetime. The projection of the five-momentum on $4D$ provides the correct definition for the rest mass. The Hamilton-Jacobi equation in $4D$ suggests we interpret the extra component of the five-momentum as the electric charge of the test particle. Finally, a conservation equation for the mass and charge follows from the five-dimensional Hamilton-Jacobi.

2.1 Line element in Kaluza-Klein theory

We will consider a general five-dimensional manifold with coordinates ξ^A ($A = 0, 1, 2, 3, 4$) and metric tensor $\gamma_{AB}(\xi^C)$. The $5D$ interval is then given by

$$dS^2 = \gamma_{AB} d\xi^A d\xi^B. \quad (1)$$

We assume that the $5D$ manifold in (1) allows us to construct, a four-dimensional hypersurface that can be identified with our $4D$ space-time. In this hypersurface we can introduce an arbitrary set of coordinates x^μ ($\mu = 0, 1, 2, 3$), which are functions of ξ^A ,

$$x^\mu = x^\mu(\xi^0, \xi^1, \xi^2, \xi^3, \xi^4). \quad (2)$$

The simplest choice is the “coordinate frame”, where the first four coordinates ξ^μ are assumed to be the coordinates of spacetime

$$x^\mu = \xi^\mu. \quad (3)$$

The corresponding spacetime basis vectors, $\hat{e}_A^{(\mu)} = \partial x^\mu / \partial \xi^A$, are

$$\begin{aligned} \hat{e}_A^{(0)} &= (1, 0, 0, 0, 0), \\ \hat{e}_A^{(1)} &= (0, 1, 0, 0, 0), \\ \hat{e}_A^{(2)} &= (0, 0, 1, 0, 0), \\ \hat{e}_A^{(3)} &= (0, 0, 0, 1, 0). \end{aligned} \quad (4)$$

¹It might be worth mentioning that, the $(4+1)$ splitting of the five-dimensional equations with non vanishing electromagnetic potentials, is definitely more involved in the case of a large extra dimension than in the classical case with compact extra dimensions [23], [39].

The vector ψ^A , orthogonal to spacetime is given by

$$\psi^A = (0, 0, 0, 0, \Phi^{-1}), \quad (5)$$

where we have set $\gamma_{44} = \epsilon\Phi^2$, so that $\gamma_{AB}\psi^A\psi^B = \epsilon$. The factor ϵ is taken to be $+1$ or -1 depending on whether the extra dimension is timelike or spacelike, respectively. Denoting $\gamma_{\mu 4} = \epsilon\Phi^2 A_\mu$, the associated basis vectors $\hat{e}_{(\mu)}^A$ are given by,

$$\begin{aligned} \hat{e}_{(0)}^A &= (1, 0, 0, 0, -A_0), \\ \hat{e}_{(1)}^A &= (0, 1, 0, 0, -A_1), \\ \hat{e}_{(2)}^A &= (0, 0, 1, 0, -A_2), \\ \hat{e}_{(3)}^A &= (0, 0, 0, 1, -A_3). \end{aligned} \quad (6)$$

They satisfy $\hat{e}_A^{(\mu)}\hat{e}_{(\nu)}^A = \delta_\nu^\mu$ and $\psi_A\hat{e}_\mu^A = 0$. Also

$$\psi_A = \epsilon\Phi(A_0, A_1, A_2, A_3, 1). \quad (7)$$

With this choice, spacetime displacements are $dx^\mu = \hat{e}_A^{(\mu)}d\xi^A = d\xi^\mu$, while displacements orthogonal to spacetime, which we will denote as dx^4 , are given by

$$dx^4 = \psi_A d\xi^A = \epsilon\Phi(d\xi^4 + A_\mu dx^\mu). \quad (8)$$

Since $d\xi^A = \hat{e}_{(\mu)}^A dx^\mu + \epsilon\psi^A dx^4$, the notation becomes more symmetrical in terms of the vector $\hat{e}_{(4)}^A = \epsilon\psi^A$, $\hat{e}_A^{(4)} = \psi_A$. Thus, $dx^M = \hat{e}_A^{(M)}d\xi^A$, $d\xi^B = \hat{e}_{(C)}^B dx^C$ and the five-dimensional metric, in the local frame, becomes

$$\hat{g}_{AB} = \hat{e}_{(A)}^C \hat{e}_{(B)}^D \gamma_{CD}. \quad (9)$$

Conversely, $\gamma_{AB} = \hat{e}_A^{(M)}\hat{e}_B^{(N)}\hat{g}_{MN}$. Therefore, the line element (1) takes the well known form

$$\begin{aligned} d\mathcal{S}^2 &= g_{\mu\nu}dx^\mu dx^\nu + \epsilon\Phi^2 (d\xi^4 + A_\mu dx^\mu)^2 = ds^2 + \epsilon(dx^4)^2, \\ g_{\mu\nu} &= \gamma_{\mu\nu} - \epsilon\Phi^2 A_\mu A_\nu, \end{aligned} \quad (10)$$

where ds is the four-dimensional interval. We note that the interpretation of the four-vector A_μ strongly relies on the signature of the extra dimension. If it is spacelike, ($\epsilon = -1$), then A_μ can be interpreted as the usual four-potential of electromagnetism. If it is timelike, ($\epsilon = +1$), this interpretation fails because we would have the wrong sign for the energy-momentum tensor of electromagnetism.

2.2 The Hamilton-Jacobi equation

2.2.1 General case

The momentum P^A of a test particle in a five-dimensional world is defined in the usual way, namely,

$$P^A = M_{(5)}U^A, \quad (11)$$

where $U^A = d\xi^A/d\mathcal{S}$ is the 5D-velocity, and $M_{(5)}$ is the 5D “mass” of the particle. Thus,

$$\gamma_{AB}P^A P^B = M_{(5)}^2 \quad (12)$$

If S denotes the action as a function of coordinates, then substituting $-\partial S/\partial \xi^A$ for P_A , we obtain the Hamilton-Jacobi equation for a test particle in $5D$

$$\gamma^{AB} \left(\frac{\partial S}{\partial \xi^A} \right) \left(\frac{\partial S}{\partial \xi^B} \right) = M_{(5)}^2, \quad (13)$$

The four-dimensional momentum $P_{(\mu)}$ is the projection of the five-dimensional quantity P_A onto the $4D$ spacetime, viz.,

$$P_{(\mu)} = \hat{e}_{(\mu)}^A P_A = -\hat{e}_{(\mu)}^A \frac{\partial S}{\partial \xi^A}. \quad (14)$$

With this definition, $P_{(\mu)}$ is invariant under arbitrary transformations of coordinates $\xi^A = \xi^A(\bar{\xi}^C)$ in $5D$, and behaves as a four-dimensional vector under $4D$ transformations $x^\mu = x^\mu(\bar{x}^\lambda)$. In addition, m_0 , the effective rest mass measured by an observer in $4D$, is invariantly defined as

$$g^{\mu\nu} P_{(\mu)} P_{(\nu)} = m_0^2. \quad (15)$$

This definition is entirely free of the problems mentioned in the introduction. Namely, the rest mass (15) is independent on the particular set of coordinates and the choice of parameter used to characterize the motion.

From (9), it follows that $\gamma^{AB} = \hat{e}_{(M)}^A \hat{e}_{(N)}^B \hat{g}^{MN}$. Using this in (13) we obtain

$$g^{\mu\nu} \left(\hat{e}_{(\mu)}^A \frac{\partial S}{\partial \xi^A} \right) \left(\hat{e}_{(\nu)}^B \frac{\partial S}{\partial \xi^B} \right) + \epsilon \hat{e}_{(4)}^A \hat{e}_{(4)}^B \frac{\partial S}{\partial \xi^A} \frac{\partial S}{\partial \xi^B} = M_{(5)}^2 \quad (16)$$

Now the substitution of (14) and (15) into (16) yields

$$m_0^2 + \epsilon \frac{q^2}{\Phi^2} = M_{(5)}^2, \quad (17)$$

where we have used the notation

$$q = P_4 = -\frac{\partial S}{\partial y}. \quad (18)$$

Here, and in the remainder of this note, we use y instead of ξ^4 , namely $y = \xi^4$. From (4), (11) and (18) it follows that

$$q = \frac{M_{(5)} \Phi u^4}{\sqrt{1 + \epsilon(u^4)^2}}, \quad (19)$$

where

$$u^4 = \frac{dx^4}{ds} = \epsilon \Phi \left(\frac{dy}{ds} + A_\mu u^\mu \right). \quad (20)$$

In order to avoid misunderstanding, we stress the fact that u^4 is not a part of the four-velocity vector, which is $u^\mu = dx^\mu/ds$. Since the displacement orthogonal to spacetime is given by $dx^4 = \hat{e}_A^{(4)} d\xi^A$ (not by $dy = d\xi^4$), it follows that u^4 characterizes the velocity, of a test particle moving in the bulk metric, orthogonal to spacetime.

A simple expression relating m_0 , the effective rest mass in $4D$, and $M_{(5)}$ can be obtained by substituting (19) into (17), namely,

$$m_0 = \frac{M_{(5)}}{\sqrt{1 + \epsilon(u^4)^2}}. \quad (21)$$

It shows that the rest mass of a particle, perceived by an observer in $4D$, varies as a result of the five-dimensional motion along the extra direction. This is similar to $m = m_0/\sqrt{1-v^2}$, for the change of mass due the motion in space. From (19) and (21) we get

$$q = m_0 \Phi u^4. \quad (22)$$

In the case where $M_{(5)} \neq 0$, equations (17) and (22) constitute a set of two linearly independent equations. They form the basis for our discussion in the next section.

2.2.2 Massless particles in $5D$

The motion of massless particles is along isotropic geodesics. Since for such geodesics $dS = 0$, from (10) it follows that

$$ds^2 = -\epsilon(dx^4)^2. \quad (23)$$

Particles with real mass (instead of imaginary mass) follow timelike geodesics, for which $ds^2 > 0$. Thus, in the case of a spacelike extra dimension, there are two physical possibilities:

- (i) If the particle has non-vanishing motion perpendicular to spacetime, then a massless particle moving in $5D$ is perceived as a massive particle by an observer in $4D$.
- (ii) If the motion in $5D$ is longitudinal to spacetime ($dx^4 = 0$), then $ds = 0$ indicating that a massless particle in $5D$, is observed as massless particle in $4D$.

In the case of a timelike extra dimension, from $m_0^2 + q^2/\Phi^2 = 0$ it follows that there is only one physical possibility, viz., $m_0 = q = 0$.

The equation of motion for massless particles is the Eikonal equation, which differs from the one of Hamilton-Jacobi in that, in the right hand side of (13) we set $M_{(5)} = 0$. Also, in (11) the derivatives $M_{(5)}d/dS$ ought to be replaced by $d/d\lambda$, where λ is the parameter along the null geodesic [40]. The effective rest mass in $4D$ is still defined by (15), and (17) yields ($\epsilon = -1$)

$$q = \pm m_0 \Phi, \quad (24)$$

which is the same as in (22), if we notice from (23) that now $u^4 = \pm 1$.

2.3 Interpretation of q

For the physical interpretation of the fifth component of P_A , or shorter of q , we note that in the coordinate frame (6), the four-momentum (14) becomes

$$P_{(\mu)} = -\frac{\partial S}{\partial x^\mu} + A_\mu \frac{\partial S}{\partial y}. \quad (25)$$

Then, the explicit form of (15), in the coordinate frame, is as follows

$$g^{\mu\nu} \left(\frac{\partial S}{\partial x^\mu} - A_\mu \frac{\partial S}{\partial y} \right) \left(\frac{\partial S}{\partial x^\nu} - A_\nu \frac{\partial S}{\partial y} \right) = m_0^2. \quad (26)$$

For a spacelike extra dimension, A_μ can be interpreted as the electromagnetic potential. On the other hand, the Hamilton-Jacobi equation for a particle with electric charge q_e moving in an electromagnetic field is

$$g^{\mu\nu} \left(\frac{\partial S}{\partial x^\mu} + q_e A_\mu \right) \left(\frac{\partial S}{\partial x^\nu} + q_e A_\nu \right) = m_0^2. \quad (27)$$

The comparison of the last two equations suggests, that for $\epsilon = -1$, we identify $P_4 = -\partial S/\partial y$ with the electric charge of the test particle, viz.,

$$q = q_e. \quad (28)$$

Thus, the motion of a test particle in the background metric, along the *spacelike* extra dimension, is perceived by an observer in $4D$ as the electric charge. This interpretation is possible because, $q = P_4$ and q_e remain invariant in all frames of reference in $4D$, i.e., they are scalars under arbitrary coordinate transformations in spacetime.

From (22), or from (24) for massless $5D$ particles, it follows that the electric charge q_e is proportional to the scalar field Φ which varies in space and time. Note that the charges of all particle species vary in the same way (so that, for example, atoms can remain always neutral [32]-[33]). This is similar to the assumption made in cosmologies with varying q_e [29]-[33], [41].

In the case of a timelike extra dimension the interpretation of P_4 in terms of four-dimensional physics is not so clear. However, there are exact solutions of the field equations of $5D$ relativity, with good physical properties and a timelike extra dimension, for which the above interpretation seems to be applicable, with no contradictions [42].

3 Effects of Large Extra Dimensions

In this section we find the general equations for the change of m_0 and q , in the presence of an electromagnetic field and scalar potential. We present the expression for the relative variation of the fine structure constant α_{em} , corresponding to the case where c and \hbar are taken to be constants. Next, we consider the $5D$ metric in gaussian normal coordinates, which allows us to isolate the effects produced by a large extra coordinate, from those induced by the electromagnetic and scalar fields. Finally, we discuss the relationship between the changes of m_0 and q .

3.1 Variation of m_0 and q

In order to find out whether the values of q and m_0 of a test particle are affected by a large extra dimension, we should find dq/ds and dm_0/ds .

First, we consider the motion of a massive $5D$ -test particle moving in an arbitrary five-dimensional background metric. If $M_{(5)} \neq 0$, then the equations (17) and (22) are linearly independent. Thus taking derivatives we find

$$\frac{dq}{ds} = m_0 \Phi \left[\frac{1}{[1 + \epsilon(u^4)^2]} \frac{du^4}{ds} + \frac{u^4}{\Phi} \frac{d\Phi}{ds} \right], \quad (29)$$

$$\frac{dm_0}{ds} = -\frac{\epsilon q}{\Phi[1 + \epsilon(u^4)^2]} \frac{du^4}{ds}, \quad (30)$$

We now need the expression for du^4/ds . It is given by equation (82) in Ref. [23], namely

$$\frac{1}{[1 + \epsilon(u^4)^2]} \frac{du^4}{ds} = \frac{1}{2\Phi} \left(\frac{\partial g_{\mu\nu}}{\partial y} \right) u^\mu u^\nu + \left(\frac{\partial A_\mu}{\partial y} + \frac{A_\mu}{\Phi} \frac{\partial \Phi}{\partial y} - \frac{1}{\Phi} \frac{\partial \Phi}{\partial x^\mu} \right) u^\mu u^4, \quad (31)$$

Substituting we get

$$\frac{dq}{ds} = \frac{1}{2} m_0 u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y} + q u^\mu \frac{\partial A_\mu}{\partial y} + \epsilon \frac{q^2}{m_0 \Phi^3} \frac{\partial \Phi}{\partial y}. \quad (32)$$

Also,

$$\frac{dm_0}{ds} = \epsilon \frac{q^2}{m_0 \Phi^3} u^\mu \frac{\partial \Phi}{\partial x^\mu} - \epsilon \frac{q}{2\Phi^2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y} - \epsilon \frac{q^2}{m_0 \Phi^2} u^\mu \left(\frac{\partial A_\mu}{\partial y} + \frac{A_\mu}{\Phi} \frac{\partial \Phi}{\partial y} \right). \quad (33)$$

It is important to mention that the above expressions, (32) and (33), are invariant under the set of “gauge” transformations

$$\begin{aligned} x^\mu &= \bar{x}^\mu, \\ y^4 &= \bar{y}^4 + f(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3), \end{aligned} \quad (34)$$

that keep the shape of the line element (10) invariant.

Let us immediately notice that, in the case of “classical” Kaluza-Klein theory, we recover some well know results. If the extra dimension is compactified, that is, if there is no dependence on the extra coordinate, then from (32) it follows that q (or P_4) is strictly constant. On the other hand (33) can be easily integrated to recover (17). Which implies that the rest mass varies as Φ^{-1} . In particular, if there is no scalar field, then m_0 is also a constant of motion.

In any other situation where the extra dimension is large, or non-compactified, it follows from (32) that $q = P_4$ will change along the trajectory of the particle. This is a totally new feature. It was missed in classical versions of Kaluza-Klein theory because of the imposition of the so-called cylinder condition. When this condition is dropped, both parameters, q and m_0 , will change along the trajectory.

If the quantity q can be identified with the electric charge q_e , then the electric charge and rest mass are not conserved separately. The conserved quantity along the motion of the particle, is now the combination $m_0^2 + \epsilon q_e^2 / \Phi^2$.

Second, we consider massless test particles moving in $5D$. For $\epsilon = +1$, such particles are observed in $4D$ as the motion of massless particles with $q = 0$. If the extra dimension is spacelike, then it follows from (24) that massless $5D$ -particles are observed in $4D$ as having effective rest mass $m_0 = |q|/\Phi$. If $q = 0$, the motion observed in $4D$ is along isotropic geodesics ($ds = 0$) in spacetime.

At this point one could ask, how can that be? If $M_{(5)} = 0$, where does the observed effective rest mass come from? In answering this question the identification of q with the electric charge is helpful. With this interpretation, the observer in $4D$ perceives the effective rest mass m_0 as being totally of electromagnetic origin, in the usual classical sense that setting $q_e = 0$ requires $m_0 = 0$ [43]-[47].

3.2 Variation of α_{em} and σ

If we assume that the speed of light c and \hbar are constants, then the variation of the fine-structure constant α_{em} is a consequence of the variation of electric charge. With the interpretation $q = q_e$, the change of α_{em} , in explicit form, can be written as

$$\frac{1}{\alpha_{em}} \frac{d\alpha_{em}}{ds} = \epsilon \left(\frac{dy}{ds} + A_\lambda u^\lambda \right)^{-1} \Phi^{-2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y} + 2u^\mu \frac{\partial A_\mu}{\partial y} + 2 \left(\frac{dy}{ds} + A_\lambda u^\lambda \right) \frac{1}{\Phi} \frac{\partial \Phi}{\partial y}, \quad (35)$$

evaluated along the trajectory of the particle. Notice that only derivatives with respect to the extra variable appear in this equation, there are no derivatives with respect to the spacetime coordinates. Thus, a large extra dimension leads naturally to a variation of the fine structure constant.

For the Thomson cross section σ we have

$$\frac{1}{2\sigma} \frac{d\sigma}{ds} = \frac{1}{\alpha_{em}} \frac{d\alpha_{em}}{ds} - \frac{1}{m_0} \frac{dm_0}{ds}. \quad (36)$$

There is no need to write its explicit expression. It is clear that σ changes even in the case of compactified extra dimension.

3.3 Relationship between the variation of q and m_0

Equations (32) and (33) are quite complicated and it is difficult to see from them if there is any connection between dq/ds and dm_0/ds . With this in mind, let us consider the class of simplified metrics

$$dS^2 = g_{\mu\nu}(x^\rho, y)dx^\mu dx^\nu + \epsilon dy^2, \quad (37)$$

which are popular in brane-world as well as in space-time-matter theories. They allow us to isolate the effects of the large extra dimension, without the details induced by the electromagnetic and scalar field. For these metrics, (32) and (33) reduce to

$$\frac{dq}{ds} = \frac{1}{2}m_0 u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y}, \quad (38)$$

and

$$\frac{dm_0}{ds} = -\epsilon \frac{1}{2} q u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y}. \quad (39)$$

Using (20) and (22), this equation can be written as

$$\frac{1}{m_0} \frac{dm_0}{ds} = -\frac{1}{2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y} \frac{dy}{ds}. \quad (40)$$

The effects of a large extra dimension on $q = P_4$ and m_0 are closely related to the extra (or fifth) force on test particles. They are proportional to the first derivatives of the metric with respect to the extra, non-compactified coordinate [21]-[25]. Indeed, although there is some controversy in the literature, this extra force is given by

$$f^\sigma = [u^\sigma u^\lambda - g^{\sigma\lambda}] u^\rho \frac{\partial g_{\lambda\rho}}{\partial y} \frac{dy}{ds}. \quad (41)$$

On the other hand, the expression (40) is *identical* to the one obtained, from different considerations [21], in the discussion of a component of the fifth force that is parallel to the four-velocity of the test particle (in that discussion (41) would be the perpendicular component).

Let us now introduce the parameter w as

$$w = \frac{1}{2} \int \left(\frac{\partial g_{\mu\nu}}{\partial y} u^\mu u^\nu \right) ds. \quad (42)$$

Equations (38) and (39) become

$$\frac{dq}{dw} = m_0, \quad (43)$$

and

$$\frac{dm_0}{dw} = -\epsilon q. \quad (44)$$

With this parameterization, q and m_0 behave as “conjugate coordinates”, satisfying the equations

$$\frac{d^2 q}{dw^2} + \epsilon q = 0, \quad (45)$$

and $d^2 m_0/dw^2 + \epsilon m_0 = 0$. These equations, with *positive* ϵ , remind of the mechanical harmonic motion. Within this analogy, q would play the role of the displacement from equilibrium ($q = 0$), m_0 the velocity, and $d^2 q/dw^2 = -\epsilon q$ the “restoring” force. The first integral of (45) would give back the conservation equation (17), i.e., $(dq/dw)^2 = M_{(5)}^2 - q^2$. Thus, q would “oscillate” with amplitude $M_{(5)}$, viz.,

$$q = M_{(5)} \sin(w - \bar{w}), \quad (46)$$

where \bar{w} is a constant of integration. Similarly, for the rest mass

$$m_0 = M_{(5)} \cos(w - \bar{w}). \quad (47)$$

Now, for a spacelike ($\epsilon = -1$) large extra dimension, the mechanical analog would correspond to an elliptical motion

$$\begin{aligned} m_0 &= M_{(5)} \cosh[w - \bar{w}], \\ q &= M_{(5)} \sinh[w - \bar{w}]. \end{aligned} \quad (48)$$

The above analogy can be formally extended to test particles moving in an arbitrary $5D$ bulk metric, with non-vanishing electromagnetic potentials and $\Phi = 1$. Indeed, from (17) it follows that one can formally set

$$\begin{aligned} m_0 &= \tilde{m}_0 \cos(\tilde{w} - \tilde{w}_0) - \tilde{q} \sin(\tilde{w} - \tilde{w}_0), \\ q &= \tilde{m}_0 \sin(\tilde{w} - \tilde{w}_0) + \tilde{q} \cos(\tilde{w} - \tilde{w}_0), \end{aligned} \quad (49)$$

for $\epsilon = +1$, and

$$\begin{aligned} m_0 &= \tilde{m}_0 \cosh(\tilde{w} - \tilde{w}_0) + \tilde{q} \sinh(\tilde{w} - \tilde{w}_0), \\ q &= \tilde{m}_0 \sinh(\tilde{w} - \tilde{w}_0) + \tilde{q} \cosh(\tilde{w} - \tilde{w}_0), \end{aligned} \quad (50)$$

for $\epsilon = -1$, provided the parameter \tilde{w} is a solution of the equation

$$\frac{d\tilde{w}}{ds} = \frac{\partial g_{\mu\nu}}{\partial y} u^\mu u^\nu + \frac{q}{m_0} u^\mu \frac{\partial A_\mu}{\partial y}. \quad (51)$$

The constants $\tilde{m}_0 = m_0(\tilde{w}_0)$ and $\tilde{q} = q(\tilde{w}_0)$ satisfy $\tilde{m}_0^2 + \epsilon \tilde{q}^2 = M_{(5)}^2$. It is clear that the above simple mechanical analog is a unique consequence of the dependence of five-dimensional metrics on the extra coordinate. This is the theoretical framework in brane-world theories and STM.

At this point we notice that in a recent paper Wesson [48] discusses five-dimensional general relativity with two “times”, i.e., in spacetimes with an extra timelike dimension. For null geodesics in $5D$, using “canonical” coordinates he shows that the parameter l , which is related to the rest mass of test particles in $4D$, oscillates in a simple harmonic motion, similar to our equations (47) and (49). Therefore, our approach can be used to extend Wesson’s discussion to massive particles in $5D$, with the advantage that here we employ an invariant definition for the rest mass m_0 .

We would like to finish this section with the following comment. Since the last term in (32) is proportional to $\partial\Phi/\partial y$, the question may arise of whether the consideration of the metric

$$d\mathcal{S}^2 = g_{\mu\nu}(x^\rho) dx^\mu dx^\nu + \epsilon \Phi^2(y) dy^2, \quad (52)$$

would not induce a spurious variation of q . This question is similar to ask whether the energy would be conserved along the motion of a test particle in a spacetime with metric $ds^2 = g_{00}(t)dt^2 + g_{ij}(x^1, x^2, x^3)dx^i dx^j$. The energy equation would be $dP_0/ds = (m_0/2)(\partial g_{00}/\partial t)(dt/ds)^2$. In this case, it is clear that the “no-conservation” of energy is not a result of any physical mechanism, but it is due to the bad choice of time coordinate. The same discussion can be applied for a charged test particle moving in the background metric (52). For this metric, Eq. (32) is equivalent to $dP_4/d\mathcal{S} = dq/d\mathcal{S} = (M_{(5)}/2)(\partial\gamma_{44}/\partial y)(dy/d\mathcal{S})^2$. Once again, the “wrong” choice of coordinate y leads to a spurious change of charge that can be eliminated by choosing a new coordinate $d\bar{y} = \Phi(y)dy$.

4 Variation of charge and rest mass in a cosmological setting

The goal of this section is twofold. Firstly, to present a simple example that illustrates the usefulness of the Hamilton-Jacobi method for the invariant definition of rest mass. Secondly, to examine the variation of charge, rest mass, α_{em} and σ in a cosmological setting.

We will consider the motion of test particles in the background metric [8]

$$dS^2 = y^2 dt^2 - t^{2/\alpha} y^{2/(1-\alpha)} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] - \alpha^2 (1-\alpha)^{-2} t^2 dy^2, \quad (53)$$

where α is a constant, y is the coordinate along the extra-dimension and t, r, θ and ϕ are the usual coordinates for a spacetime with spherically symmetric spatial sections. This is a solution to the five-dimensional Einstein field equations, with ${}^{(5)}T_{AB} = 0$.

In four-dimensions (on the hypersurfaces $y = \text{const.}$) this metric corresponds to the 4D Friedmann-Robertson-Walker models with flat 3D sections. The energy density ρ_{eff} and pressure p_{eff} of the effective 4D matter satisfy the equation of state

$$p_{eff} = n \rho_{eff}, \quad (54)$$

where $n = (2\alpha/3 - 1)$. Thus for $\alpha = 2$ we recover radiation, for $\alpha = 3/2$ we recover dust, etc.

In spherically symmetric fields test particles move on a single “plane” passing through the center. We take this plane as the $\theta = \pi/2$ plane. Then, the Hamilton-Jacobi equation, for the metric (53) is

$$\frac{1}{y^2} \left(\frac{\partial S}{\partial t} \right)^2 - \frac{1}{t^{2/\alpha} y^{2/(1-\alpha)}} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \phi} \right)^2 \right] - \frac{(1-\alpha)^2}{\alpha^2 t^2} \left(\frac{\partial S}{\partial y} \right)^2 = M_{(5)}^2. \quad (55)$$

It is clear that the action separates as

$$S = S_1(t, y) + S_2(r) + L\phi, \quad (56)$$

where L is the angular momentum. Thus, we obtain

$$\frac{1}{y^2} \left(\frac{\partial S_1}{\partial t} \right)^2 - \frac{k^2}{t^{2/\alpha} y^{2/(1-\alpha)}} - \frac{(1-\alpha)^2}{\alpha^2 t^2} \left(\frac{\partial S_1}{\partial y} \right)^2 = M_{(5)}^2, \quad (57)$$

and

$$\left(\frac{dS_2}{dr} \right)^2 + \frac{L^2}{r^2} = k^2 \geq 0, \quad (58)$$

where k is the separation constant. If it is zero, then the particle is commoving (or at rest) in the system of reference defined by (53).

In this case, the spacetime basis vectors (6) reduce to $\hat{e}_{(\mu)}^A = \delta_\mu^A$. Thus, from (14) and (15) we obtain the effective rest mass, as measured in 4D, as follows

$$m_0^2 = \frac{1}{y^2} \left(\frac{\partial S_1}{\partial t} \right)^2 - \frac{k^2}{t^{2/\alpha} y^{2/(1-\alpha)}}, \quad (59)$$

evaluated at the trajectory $y = y(t)$. We stress the fact that this expression is totally free of the ambiguities, induced by the choice of parameters used to characterize the motion in 5D and 4D, typical of the approach where the equations of motion for test particles in 4D are obtained from the geodesic equation in 5D.

The relative variation of m_0 , for particles fixed in space $u^\mu = \delta_0^\mu / y$, is given by

$$\frac{1}{m_0} \frac{dm_0}{ds} = -\frac{1}{y} \frac{dy}{ds} \left[1 + \frac{\alpha^2 t}{(1-\alpha)^2} \frac{dy}{ds} \right]. \quad (60)$$

Also, for the fine structure constant we get

$$\frac{1}{\alpha_{em}} \frac{d\alpha_{em}}{ds} = -\frac{2(1-\alpha^2)^2}{\alpha^2 t^2 y} \left(\frac{dy}{ds} \right)^{-1}. \quad (61)$$

These quantities have to be evaluated along the particle's worldline. The variation of σ is then obtained from (36).

At this point, it is worth noticing that in the case where $q = P_4 = -(\partial S/\partial y) = 0$, the equation of motion (57) requires $y = \text{const}$, and the effective rest mass observed in $4D$ is $m_0 = M_{(5)}$. In other words a particle moving in $5D$ with no momentum in the extra direction, which is observed in $4D$ as the motion of a particle with no-electric charge, will be insensitive to the extra dimension. Interesting results appear when the $5D$ motion has nonzero P_4 -component.

Let us now consider different physical scenarios allowed in (57)

4.1 Massive test particles: $M_{(5)} \neq 0$

In this case there are two possibilities. Either the particle is at rest ($k = 0$), or it is somehow moving in spacetime ($k \neq 0$). In order to isolate the effects from the extra dimension, from the effects due to the motion in spacetime, we will consider $k = 0$.

From (57) one can easily get

$$S_1(t, y) = \pm \frac{\alpha M_{(5)}}{\sqrt{2\alpha-1}} yt. \quad (62)$$

Consequently,

$$m_0 = \frac{\alpha M_{(5)}}{\sqrt{2\alpha-1}}, \quad q = \pm m_0 t. \quad (63)$$

Here the rest mass is constant, because of the mutual cancelation of the change induced by the term $(\partial g_{\mu\nu}/\partial y)u^\mu u^\nu$ and the change induced by the scalar field.

Now, using

$$P^A = M_{(5)} \frac{d\xi^A}{d\mathcal{S}} = \gamma^{AB} P_B = -\gamma^{AB} \frac{\partial S}{\partial \xi^B}, \quad (64)$$

we obtain

$$\frac{dt}{d\mathcal{S}} = \mp \frac{\alpha}{y\sqrt{2\alpha-1}}, \quad (65)$$

and

$$\frac{dy}{d\mathcal{S}} = \pm \frac{(1-\alpha)}{t\sqrt{2\alpha-1}}. \quad (66)$$

From these expressions we evaluate dy/dt and integrate to obtain

$$y = Dt^{(\alpha-1)/\alpha}, \quad (67)$$

along the trajectory. Here D is a constant of integration. The proper time τ along the trajectory is given by $d\tau = y(t)dt$. Thus, from (67), we get $\tau \sim t^{(2\alpha-1)/\alpha}$. Consequently, in this case, m_0 is a constant and $q \sim \tau^{\alpha/(2\alpha-1)}$.

Since m_0 is constant here, it follows from (36) that the change of the Thomson cross section σ is similar to that of α_{em} , which is given by

$$\frac{1}{\alpha_{em}} \frac{d\alpha_{em}}{d\tau} = \frac{2\alpha^2}{(2\alpha-1)\tau}. \quad (68)$$

Models with $\alpha < 1$ have inflationary equations of state. During the cosmological “constant” era evolution $p_{eff} \approx -\rho_{eff}$ ($\alpha \rightarrow 0$), α_{em} and σ decrease with the increase of τ . Then, in the curvature era, and further in the radiation and dust dominated era α_{em} and σ grow as a power of τ .

4.2 Massless test particles: $M_{(5)} = 0$

In this case the trajectory in $5D$ is along isotropic geodesics. These are given by the Eikonal equation, which is formally obtained from the above formulae by setting $M_{(5)} = 0$ in (57). According to the discussion in section 2.2.2, there are two different physical possibilities here. They are $q \neq 0$, or $q = 0$.

4.2.1 $M_{(5)} = 0$, $q \neq 0$

For the same reason as above, we will consider $k = 0$. Then, equation (57) separates and, we obtain

$$S_1 = Ct^{\pm l} y^{\pm l\alpha/(1-\alpha)}, \quad (69)$$

where C is a constant of integration and l is the separation constant. Now in (64) instead of the derivatives $M_{(5)}d/dS$ we have to write derivatives $d/d\lambda$, where λ is the parameter along the null geodesic. Thus, we find

$$\frac{dt}{d\lambda} = \mp Clt^{(-1\pm l)} y^{(-2\pm l\alpha/(1-\alpha))}, \quad (70)$$

and

$$\frac{dy}{d\lambda} = \pm Cl \frac{1-\alpha}{\alpha} t^{(-2\pm l)} y^{(-1\pm l\alpha/(1-\alpha))}. \quad (71)$$

From the above expressions we evaluate dy/dt and integrate to obtain $y = Dt^{(\alpha-1)/\alpha}$. This is the same equation as in the massive case (67). However, in this case we find q and m_0 as follows

$$\begin{aligned} q &= \mp A\alpha(1-\alpha)^{-1} t^{(1-\alpha)/\alpha}, \\ m_0 &= At^{(1-2\alpha)/\alpha}, \end{aligned} \quad (72)$$

where A is expressed through the other constants as $A = ClD^{(-1\pm l\alpha/(1-\alpha))}$. Thus, here $q \sim \tau^{(1-\alpha)/(2\alpha-1)}$ and $m_0 \sim \tau^{-1}$. Thus,

$$\frac{1}{\sigma} \frac{d\sigma}{d\tau} = \frac{2}{(2\alpha-1)\tau}, \quad \frac{1}{\alpha_{em}} \frac{d\alpha_{em}}{d\tau} = \frac{2(1-\alpha)}{(2\alpha-1)\tau}. \quad (73)$$

Near the vacuum dominated era $p_{eff} \approx -\rho_{eff}$, α_{em} and σ decrease with the increase of τ . In the quintessential epoch for which $-2\rho_{eff}/3 < p_{eff} < -\rho_{eff}/3$ ($1/2 < \alpha < 1$), α_{em} and σ grow with time. In the very early universe ($\alpha = 3$), radiation epoch ($\alpha = 2$) and dust era ($\alpha = 3/2$), α_{em} decreases and σ increases as a power of the proper time τ .

Hence there are very different possibilities for the change in α_{em} or q_e depending on whether the test particles in $5D$ are assumed massive ($M_{(5)} \neq 0$) or massless ($M_{(5)} = 0$). However, the Thomson cross section σ has a similar behavior in both cases, except in the quintessential epoch.

4.2.2 $M_{(5)} = 0$, $q = 0$

In this case the motion, as observed in $4D$, is lightlike and, therefore, k must be different from zero. The left hand side in (59) is $m_0 = 0$. This equation is equivalent to $k^\mu k_\mu = 0$, where k_μ is the $4D$ wave vector. Since $P_4 = 0$, it follows that $y = \text{const}$ along the motion. Therefore, the frequency $\omega = -\partial S/\partial t$ of the “induced” photon is

$$\omega \sim \tau^{-1/\alpha}. \quad (74)$$

For completeness, we mention that there is one more possibility left here. Namely the motion in $5D$ with $dS^2 < 0$. In this case replacing $M_{(5)}^2 \rightarrow -\bar{M}^2$ we obtain the same expressions as in section 4.1, but with \bar{M} instead of $M_{(5)}$ and $\sqrt{1-2\alpha}$ instead of $\sqrt{2\alpha-1}$. Thus, such a five-dimensional motion is interpreted by an observer in $4D$ as a test particle with positive effective rest mass. This is a pure consequence of the motion in $5D$ along the *spacelike* extra coordinate.

5 Summary and Conclusions

We have given the general formula for the change of the rest mass of a test particle, as measured by an observer in $4D$. Equation (33) generalizes to an arbitrary five-dimensional metric the particular expression (40) for the metric in gaussian normal coordinates (37). The variation of rest mass is predicted in classical versions of compactified Kaluza-Klein theory with scalar field. The last three terms in (33) are the “corrections” induced by a large extra dimension.

However, the electric charge is strictly constant in compactified versions of Kaluza-Klein theory. This is because the cylinder condition requires the extra coordinate y to be cyclic. Then, according to the Hamilton-Jacobi formalism, the action separates as $S = S(x^\mu) - qy$, where $q = P_4 = -\partial S/\partial y = \text{const}$. When this condition is dropped, $q = P_4$ will vary along the trajectory according to (32). Thus, if q is identified with the electric charge, then an observer in $4D$ will find “inconsistencies” with the law of conservation of charge in $4D$. In the present scenario the conserved quantity along the trajectory is the combination $m_0^2 + \epsilon q_e^2/\Phi^2$.

The variation of m_0 and q_e has important implications in a wide variety of phenomena. In particular, in the scattering of electromagnetic radiation by charged particles, for which the cross section is proportional to q_e^4/m_0^2 . Also in the variation of the fine structure constant α_{em} , recently observed in the study of quasar absorption line spectra.

The discussion in section 4 shows how to obtain the rest mass and q (or charge), as observed in $4D$, from the Hamilton-Jacobi formalism in $5D$. It also illustrates the different physical scenarios perceived by an observer in $4D$. We have seen that the effective $4D$ quantities depend not only on the motion in $5D$ but also on whether the $5D$ test particle is massive or massless. The results obtained for m_0 , q_e , α_{em} and σ are intended to be illustrative rather than experimental suggestions.

The question may arise of whether the predicted variation of rest mass and charge contradicts some well established assumptions in physics, like charge conservation. In our interpretation, the answer to this question is negative. Indeed, since the variation of these quantities takes place on cosmic timescales, they would not be observed directly in laboratory. This seems to be a general feature of constants in higher dimensional theories whose effective value in $4D$ may vary in time and space. This is similar to what happens in cosmological models that are free of singularities in $5D$, but singular when interpreted in $4D$ [49].

Our analysis also clarifies the $4D$ interpretation of null geodesics in $5D$. We have seen that such geodesics appear as timelike paths in $4D$ only if the following two conditions are met *simultaneously*: (i) the extra dimension is spacelike, and (ii) the particle in its five-dimensional motion has $P_4 \neq 0$ (or $q \neq 0$). Otherwise, a null geodesic in $5D$ is observed as a lightlike particle in $4D$.

We notice that in our discussion the underlying physics motivating the introduction of a large extra dimension was nowhere used. Neither, the physical meaning of the extra coordinate. Therefore, our results are applicable to brane-world models, STM, and other $5D$ theories with a large extra dimension².

²In STM and in the thick brane scenario the $5D$ manifold is smooth everywhere and there are no defects. In the RS2 brane-world scenario [5] our universe is a singular $4D$ hypersurface and the derivatives $\partial g_{\mu\nu}/\partial y$ are discontinuous, and change sign, through the brane. However, the discontinuity is not observed [25] and effective $4D$ equations can be obtained by taking mean values and applying Israel’s junction conditions through the brane [22].

Thus, the possibility that our $4D$ universe is embedded in a higher dimensional bulk space, with more than four non-compact extra dimensions, should have major consequences for astrophysics and cosmology. The existence of an extra (fifth) force is one of them. Here we presented the variation of the fundamental electron charge. Elsewhere we discussed the variation of the gravitational coupling G and the cosmological term $\Lambda_{(4)}$ [50]. In principle, these effects are observable, but theoretical predictions are model-dependent. Therefore, other $5D$ metrics have to be studied and tested experimentally for compatibility with observational data.

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